Math 246A Lecture 6 Notes

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1 Möbius Transformations and the Cross Ratio

1.1 Generating Möbius transformations

Theorem 1.1. *Translation, rotation, dilation, and inversion generate MT.*

Proof. Let $Tz = (az+b)/(cz+d)$.

Case 1: $c = 0$. Then $Tz = \frac{a}{d}(z + \frac{b}{a})$, which is a translation, then a dilation, then a rotation.

Case 2: $c \neq 0$. Then

$$Tz = \frac{az+b}{c(z+d/c)} = \frac{a(z+d/c)+(b-ad/c)}{c(z+d/c)} = \frac{bc-ad}{c^2(z+d/c)} + \frac{a}{c},$$

which is a translation, then an inversion, then a dilation, then a rotation, and finally another translation. $\square$

Theorem 1.2. *Assume z_2, z_3, z_4 are distinct points in $\mathbb{C}^*$. Then there exists a unique $S \in MT$ such that*

$$S(z_2) = 1, S(z_3) = 0, S(z_4) = \infty.$$

Proof. If $z_2 = \infty$, we get

$$\frac{z-z_3}{z-z_4}$$

If $z_3 = \infty$, then we get.

$$\frac{z_2-z_4}{z-z_4}.$$

If $z_4 = \infty$, then we get

$$\frac{z-z_3}{z_2-z_3}.$$

If none of them equal ∞, then we get

$$\frac{z-z_3}{z-z_4}\bigg/\frac{z_2-z_3}{z_2-z_4}.$$

$\square$

Remark 1.1. This property is called being **triply transitive**.

1.2 The cross ratio and correspondence of circles and lines

Definition 1.1. The **cross ratio** (z_1, z_2, z_3, z_4) of four distinct points is $S(z_1)$, where $S(z_2) = 1, S(z_3) = 0, S(z_4) = \infty$.

Proposition 1.1. *$(Tz_1, Tz_2, Tz_3, Tz_4) = (z_1, z_2, z_3, z_4)$ for all Möbius transformations T.*

Proof. Let the right hand side be $S(z_1)$. Then $S \circ T^{-1}(Tz_i) = S(z_i)$, and the left hand side is $S \circ T^{-1}$. $\square$

Theorem 1.3. *Let z_1, z_2, z_3, z_4 be distinct. Then*

1. *$(z_1, z_2, z_3, z_4) \in \mathbb{R} \cup \{\infty\} \iff \{z_1, z_2, z_3, z_4\} \subseteq C$ for a circle $C \subseteq C^*$*
2. *$T \in MT \implies T^{-1}(\mathbb{R} \cup \{\infty\})$ is a circle in C^*.*

Proof. Proof 1 of part 1: Let z, z_2, z_3, z_4 be on a circle. Then

$$(z, z_2, z_3, z_4) = -\frac{z - z_3}{z - z_4} \Big/ -\frac{z_2 - z_3}{z_2 - z_4}.$$

The top part has argument $\angle z_3 z z_4$, and the bottom has argument $\angle z_1 z z_4$. By geometry, these are equal, so $\arg((z, z_2, z_3, z_4)) = 0$.

Proof 2 of part 1: Let $T = (aw + b)/cw + d)$. Then

$$\begin{aligned} w \in T^{-1}(\mathbb{R}) &\iff T(w) = \overline{T(w)} = (\overline{a}\overline{w} + \overline{b})/(\overline{c}\overline{w} + \overline{d}) \\ &\iff (aw + b)(\overline{c}\overline{w} + \overline{d}) = (cw + d)(\overline{a}\overline{w} + \overline{b}) \\ &\iff (a\overline{c} - \overline{a}c)|w|^2 + (a\overline{d} - \overline{b}c)w + (b\overline{c} - \overline{a}d)\overline{w} + (b\overline{d} - \overline{b}d) = 0 \end{aligned}$$

Case 1: $a\overline{c} - \overline{a}c = 0$. This is a line.
Case 2: $a\overline{c} - \overline{a}c = 0$. Then

$$Tw = \overline{Tw} \iff |w|^2 - \frac{\overline{a}d - \overline{b}c}{a\overline{c} - \overline{a}c}w - \frac{a\overline{d} - b\overline{c}}{a\overline{c} - \overline{a}c}\overline{w} + \frac{b\overline{d} - \overline{b}d}{a\overline{c} - \overline{a}c} = 0$$

$$\left| w - \frac{\overline{a}d - \overline{b}c}{a\overline{c} - \overline{a}c} \right| = |w|^2 - \frac{\overline{a}d - \overline{b}c}{a\overline{c} - \overline{a}c}w - \frac{\overline{a}d - b\overline{c}}{a\overline{c} - \overline{a}c}\overline{w} + \frac{b\overline{d} - \overline{b}d}{\overline{a}c - a\overline{c}} = 0.$$

So

$$\left| w - \frac{b\overline{c} - \overline{a}d}{\overline{a}c - a\overline{c}} \right|^2 = R^2$$

for some $R \in \mathbb{R}$.

To prove part 2, look at $T(z) = (z - i)/(z + i)$. If $z \in \mathbb{R}$, then $|Tz| = 1$, so T maps $\mathbb{R} \cup \{\infty\}$ to the unit circle. $\square$

1.3 Symmetric points

Definition 1.2. Let C be a circle and $T : \mathbb{R} \to C$. Points z, z^* are **symmetric w.r.t.** C if $T^{-1}(z^*) = \overline{T^{-1}(z)}$. Points w, w^* are **symmetric w.r.t.** $\mathbb{R}$ if Tw, Tw^* are symmetric w.r.t. C for any T such that $T(\mathbb{R}) = C$.

Example 1.1. If C is the unit circle, the points $z, 1/\overline{z}$ are symmetric w.r.t. C.

Theorem 1.4. *Suppose $T(C_1) = T(C_2)$. Then z, z^* are symmetric w.r.t. C_1 iff Tz, Tz^* are symmetric w.r.t. C_2.*